Intro to Rep theory → RT 1 (BA 5)
→ RT 2 (BA 6)
→ RT 3.a) and 3.b) (MA 2)

group $G \curvearrowright$ vector space V

<u>Def</u>: let V be a $\mathbb{F}$-vector space

a representation $G \curvearrowright V$ is an assignment

$\forall g \in G \rightsquigarrow \Phi_g : V \to V$ a $\mathbb{F}$-linear function

satisfying $\Phi_e = Id_V$, $\Phi_{g^{-1}} = \Phi_g^{-1}$, $\Phi_{gh} = \Phi_g \circ \Phi_h$.

$$\Phi_g(\alpha v + \alpha' v') = \alpha \Phi_g(v) + \alpha' \Phi_g(v')$$
$$\forall \alpha, \alpha' \in \mathbb{F}, \ \forall v, v' \in V$$

Ex: $D_{2n} \curvearrowright \mathbb{R}^2$ is a representation

φ rot, refl acts by itself

Assume $\dim V = n \rightsquigarrow V \cong \mathbb{F}^n$ (choosing a basis)

$$\mathbb{F}^n = \left\{ v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \;\middle|\; x_1, \dots, x_n \in \mathbb{F} \right\}$$

any $\mathbb{F}$-linear $\Phi : V \to V$ takes the form

$\Phi(v) = A v$ for some $n \times n$ matrix A with coeffs in $\mathbb{F}$

$$G\text{-representation} \rightsquigarrow \{\Phi_g : V \to V\}_{g \in G} \rightsquigarrow \{A_g \in \mathrm{Mat}_{n\times n}(\mathbb{F})\}_{g \in G}$$

s.t. $A_e = I_n$

$A_{g^{-1}} = A_g^{-1}$

$A_{gh} = A_g A_h$

homomorphism

$G \longrightarrow GL_n(\mathbb{F})$

multiplicative group of invertible $n \times n$ matrices with coeffs in $\mathbb{F}$

<u>Def</u>: $G \curvearrowright V$; $G \curvearrowright W$ a **G-intertwiner** is a $\mathbb{F}$-linear $f : V \to W$ which commutes with the G-action

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "g\cdot"'] \arrow[dr, phantom, "\square"] & W \arrow[d, "g\cdot"] \\ {} & {} \end{tikzcd} \quad \forall g \in G$$

Notation: $g \cdot v = \Phi_g(v)$

$$V \xrightarrow{f} W$$

i.e. $f(g\cdot v) = g \cdot f(v), \forall v \in V$

A bijective G-intertwiner is called an isomorphism.

<u>Def</u>: $G \curvearrowright V$; then a linear subspace $W \subset V$ is called a subrepresentation (of V) if it is preserved by the G-action, i.e.

$$g \cdot W \subseteq W, \quad \forall g \in G$$

The quotient representation is $V/_W$ with the action

$$g \cdot (v \bmod W) = g \cdot v \bmod W$$

<u>Def</u>: a representation $G \curvearrowright V$ is irreducible if it has no proper subrepresentation (other than 0 and V)

Schur's lemma: suppose you have an intertwiner

$$V \xrightarrow{f} W$$

of representations of G, suppose f not identically 0

- if V is irreducible, then f is injective
- if W is irreducible, then f is surjective
- if V and W are irreducible, then f is an isomorphism

Proof: $\operatorname{Ker} f$ is a subrepresentation of V

$v \in \operatorname{Ker} f$, $\forall g \in G$, $f(g \cdot v) = g \cdot f(v) = g \cdot 0 = 0 \Rightarrow g \cdot v \in \operatorname{Ker} f$

$\operatorname{Im} f$ is a subrepresentation of W

$w \in \operatorname{Im} f$, $\forall g \in G$, $\exists\, v \in V$ s.t. $w = f(v)$

$g \cdot w = g \cdot f(v) = f(g \cdot v) \in \operatorname{Im} f$

$\mathbb{F} = \mathbb{C}$

Prop: if $V \cong \mathbb{C}^n$ is an irreducible representation of G

then the only intertwiners $f: V \to V$ are scalars $z \cdot Id_V$

if $V \cong W$, all isomorphisms $V \to W$ are multiples of each other

($f_1, f_2 : V \to W$, then $f_2 \circ f_1^{-1} : V \to V$ is an intertwiner, so $f_2 \circ f_1^{-1} = z \cdot Id_V$)

Proof: the intertwiner $f: V \to V$ has an eigenvector $0 \neq v \in V$,

i.e. $f(v) = z v$ for some $z \in \mathbb{C}$

Look at $W = \{ w \in V \mid f(w) = z w \} \neq 0$

Claim: W is a subrepresentation of V

if $w \in W$, $\forall g \in G$, $f(g \cdot w) = g \cdot f(w) = g \cdot z w$
$= z\, g \cdot w \Rightarrow g \cdot w \in W$

$W = V$ by irreducibility $\Rightarrow f = z\, \mathrm{Id}_V$ $\square$

Def: $G \curvearrowright V$, $G \curvearrowright W$, the direct sum is $G \curvearrowright V \oplus W$ with action $g \cdot (v, w) = (g \cdot v, g \cdot w)$

Maschke's Theorem: any finite-dimensional complex representation $G \curvearrowright V$, where G is a finite group, is completely reducible

$V \cong V_1 \oplus \cdots \oplus V_k$ where $V_1, \dots, V_k$ are irreducible.

$V = V_1 \oplus \cdots V_2$

Ex: $G = \mathbb{Z}/2\mathbb{Z} \ni \{0,1\} \rightsquigarrow A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{C})$

$V = \mathbb{C}^2$ $\qquad A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in GL_2(\mathbb{C})$

A subrepresentation is a subspace of $\mathbb{C}^2$ which is preserved by A_1

$$\mathbb{C}\begin{pmatrix} x \\ y \end{pmatrix}, \quad A_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$$

A_1 preserves only two lines: diagonal $\mathbb{C}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

: anti-diagonal $\mathbb{C}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$V = \mathbb{C}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cong \mathbb{C}_{triv} \oplus \mathbb{C}_{sign}$$

of $\mathbb{Z}/2\mathbb{Z}$ -representations

$A_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $\mathbb{C}_{triv} = \mathbb{C}$ with the trivial $\mathbb{Z}/2\mathbb{Z}$-action
$A_0 = (1)$, $A_1 = (1)$

$A_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -\begin{pmatrix} 1 \\ -1 \end{pmatrix}$; $\mathbb{C}_{sign} = \mathbb{C}$ with the non-trivial $\mathbb{Z}/2\mathbb{Z}$-action
$A_0 = (1)$, $A_1 = (-1)$

Cool fact: up to isomorphism, the number of irreducible representations of any finite group G is the number of conjugacy classes of G.

(e.g. $G = S_n$, the number of irreducible representations of S_n = the number of partitions of n)

Maschke's theorem fails for infinite groups

$$(\mathbb{R}, +) \curvearrowright \mathbb{R}^2$$

$$x \rightsquigarrow \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$, because $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix}$

$\mathbb{R}^2$ is not irreducible, because it has $\mathbb{R} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as its unique subrepresentation $\Rightarrow \nexists$ lines l_1 and l_2 s.t. $\mathbb{R}^2 \cong l_1 \oplus l_2$

Maschke: a finite group G

$\{L_1, \ldots, L_d\}$ is a complete family of irreducible representations of G up to iso ($d = \#$ of conjugacy classes of G)

any $G \curvearrowright V$ has the property that

$$V \cong L_1^{\oplus n_1} \oplus L_2^{\oplus n_2} \oplus \cdots \oplus L_d^{\oplus n_d}$$

where $n_1, \ldots, n_d$ are called the multiplicities of $L_1, \ldots, L_d$ inside V and they can be calculated by character theory.

$$\begin{tikzcd} V \arrow[d, "\phi"'] & \cong & L_1^{\oplus n_1} \arrow[d, "\phi_1"] & \oplus \cdots \oplus & L_d^{\oplus n_d} \arrow[d, "\phi_d"] \\ W & \cong & L_1^{\oplus m_1} & \oplus \cdots \oplus & L_d^{\oplus m_d} \end{tikzcd}$$

intertwiner

Schur's Lemma says no intertwiners $L_i \to L_j$ if $i \neq j$

each $\phi_i : L_i^{\oplus n_i} \longrightarrow L_i^{\oplus m_i}$; only intertwiners $L_i \to L_i$ are scalar multiples of Id_{L_i}

$$\phi_i\left(\begin{pmatrix} v_1 \\ \vdots \\ v_{n_i} \end{pmatrix}\right) = \begin{pmatrix} \gamma_{11} v_1 + \gamma_{12} v_2 + \cdots + \gamma_{1n_i} v_{n_i} \\ \vdots \\ \gamma_{m_i 1} v_1 + \cdots + \gamma_{m_i n_i} v_{n_i} \end{pmatrix} \quad \forall\, v_1, \ldots, v_{n_i} \in L_i$$

ϕ_i is completely determined by $\begin{pmatrix} \gamma_{11} & \cdots & \gamma_{1n_i} \\ \vdots & & \vdots \\ \gamma_{m_i 1} & \cdots & \gamma_{m_i n_i} \end{pmatrix} \in Mat_{m_i \times n_i}(\mathbb{C})$

All intertwiners are completely classified by providing a d-tuple of matrices, whose sizes are dictated by the multiplicities